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Optimal Controller Switching For Stochastic Systems

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Abstract. This paper presents a solution to certain problems in switched controller design for stochastic dynamical systems with a quadratic cost. The main result is a separation theorem for partial information systems. This result is then used to convert the partial information stochastic control problem to a complete information stochastic control problem. We also show that certainty equivalence does not hold. The optimal sequence of controllers can be determined via an appropriate solution to a dynamic programming problem.

1 Introduction

There are many real industrial control problems where the control action is determined by switching among a finite number of given control laws. Typical examples include an automobile transmission, where different gears need to be switched in order to obtain desirable performance (gain switching) [1, 2], a thermostat where the plant heats up when the control is on and cools when the control is off, and steering an automobile where switching between appropriately chosen controls laws can make an otherwise uncontrollable vehicle controllable and allow the vehicle to be steered (locally) in any desired direction [3](pg 76).

In [4, 5] we devised optimal strategies for switching between given controllers at fixed switching intervals for plants described by an uncertain differential equation. In this paper we consider optimal switching strategies for plants described by linear time-varying stochastic differential equations with a quadratic cost

functional and develop optimal switching strategies for both the full and partial information cases [6,7]. We consider separation and certainty equivalence for switched systems. Although these notions have been studied for many years [8] and their applicability to various classes of stochastic systems continues to be of great interest e.g. [9]. To our knowledge these important notions have not been investigated for linear switched controller and hybrid control systems. This paper presumes a background knowledge of linear stochastic control systems and Ito calculus. An excellent treatment of these subjects can be found in [10] and [11].

In this paper we show that the full information problem with quadratic cost the optimal switching sequence of controllers is given by a solution to a dynamic programming equation. We show that where the switching function depends on both time and state (state estimate) certainty equivalence does not hold. For the partial information case a separation theorem is established. This is then used to convert the partial information problem to a full information problem enabling the optimal controller switching sequence to be determined.

The organization of this paper is as follows. Section 2 considers full information switched controller systems. The main results of this section is to show equivalence between the optimal controller switching sequence and the solution to a dynamic programming equation. In this section it is also shown that for the class of systems considered in this paper certainty equivalence does not hold. In section 3 the partial information problem is considered. The main results of this section are given in terms of the existence of suitable solutions to a Ricatti differential equation of the Kalman Filtering type and appropriate solutions to a dynamic programming equation. In section 4 a simulation example is presented.

2 Optimal Controller Switching with Full Information

All random variable in this paper are defined on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Let $w(\cdot)$ be a vector valued Wiener process with covariance matrix $Q(\cdot)$. Let $\mathcal{E}$ denote expectation with respect to the measure $\mathcal{P}$. Let $N \in \mathbf{N}^+$ be a given positive integer. The permissible switching times $\{t_0, t_1, t_2, \ldots, t_N\}$ corresponding to an ordered discrete set where $(t_i \in \mathbf{R}^+)$.

Consider the following stochastic system whose dynamics on the interval $[0, t_N)$ are given by the stochastic differential equation

$$dx(t) = (A(t)x(t) + B_2(t)u(t))dt + B_1(t)dw(t); \quad x(0) = x_0 \qquad (1)$$

where $x(t) \in \mathbf{R}^n$ denotes the state, $u(t) \in \mathbf{R}^h$ is the control and the functions $A(t), B_1(t), B_2(t), C(t)$ are given piecewise continuous matrix functions of time of appropriate dimension.

Controller Switching with State Feedback [12,5,4,13]. Given a finite collection of controllers

$$u_1(t) = U_1(t, x(t)), \quad u_2(t) = U_2(t, x(t)), \quad \ldots, \quad u_k(t) = U_k(t, x(t)) \qquad (2)$$

where the control laws $U_1(\cdot,\cdot), U_2(\cdot,\cdot), \ldots, U_k(\cdot,\cdot)$ are given continuous matrix functions. Let $I_j(\cdot)$ denote a state switching function which is permitted to switch between controllers at set switching times t_j, $\{j = 1, \ldots, N\}$. The class of switching functions considered in this section is of the form $I_j(\cdot) : \{x(\cdot) \mid_0^{t_j}\} \to \{1, 2, \ldots, k\}$, which map from the set of state measurements $\{x(\cdot) \mid_0^{t_j}\}$ to the set of symbols $\{1, 2, \ldots, k\}$ which index the controllers (2). This class of switching functions defines a family of dynamic nonlinear state feedback controllers of the form:

$$\forall j \in \{0, 1, \ldots, N-1\} \ \ u(t) = U_{i_j}(t, x(t)) \ \forall t \in [t_j, t_{(j+1)}) \ \ where \ i_j = I_j(x(\cdot) \mid_0^{t_j}). \tag{3}$$

Hence the control strategy, which is a rule for switching from one controller to another, constructs a symbolic sequence $\{i_j\}_{j=0}^{N-1}$ from the set of state measurements $x(\cdot) \mid_0^{t_j}$. The set of all admissible controls achievable by controller switching (3) with the controllers (2) is denoted by $\mathcal{U}$.

Let $S(\cdot)$ be a given function which maps from R^n to R, let $N(t)$ and $M(t)$ be given matrices of time satisfying $N(t), M(t) \geq \rho I$, $\rho > 0 \ \forall t \in [0, t_N]$. $X_f \geq 0$ is a given constant matrix which penalises the final state.

Introduce the following quadratic cost function,

$$W(t, x, u(t)) \triangleq \mathcal{E}\left[x'(t)N(t)x(t) + u'(t)M(t)u(t)\right] \tag{4}$$

and define

$$F_j^i(x_0, S(\cdot)) \triangleq \mathcal{E}[S(x(t_{j+1}))] + \int_{t_j}^{t_{j+1}} W(t, x, u_i(t))dt. \tag{5}$$

Definition 1 *(State Feedback Stochastic Control Problem). The state feedback stochastic control problem with system (1) is said to have a solution via controller switching with the controllers (2) if the following conditions hold:*

(i) For any admissible control sequence, $u \in \mathcal{U}$, there exists a unique trajectory of system (1) on the interval $[0, t_N)$.
(ii) The cost

$$J(u)_{u \in \mathcal{U}} = \mathcal{E}\left(\int_0^{t_N} x'(t)N(t)x(t) + u'(t)M(t)u(t)dt + x'_{t_N}X_f x_{t_N}\right) \tag{6}$$

exists and is finite for some admissible control sequence, $u \in \mathcal{U}$.

Theorem 1. *The state feedback stochastic control problem (Definition (1)) for system (1) has a solution via controller switching with the state feedback controllers (2) if and only if the dynamic programming problem*

$$V_N(x_0) = \mathcal{E}\left[x'_{t_N}X_f x_{t_N}\right]; \ \ V_j(x_0) = \min_{i=1,\ldots,k} \mathcal{E}\left[F_j^i(x_0, V_{j+1}(\cdot))\right] \tag{7}$$

has a solution for all $j = 0, 1, \ldots, N-1$ and all $x_0 \in \mathbf{R}^n$.

Furthermore let $i_j(x_0)$ be an index such that the minimum in (7) is achieved for $i = i_j(x_0)$. Then the controller (2), (3) associated with the switching sequence $\{i_j\}_{j=0}^{N-1}$ where $i_j \triangleq i_j(x(t_j))$ solves the state feedback stochastic control problem.

Proof. Sufficiency. For any $j = 0, 1, \ldots, N$ introduce the cost to go

$$V_j \triangleq \min_{u(t) \in \mathcal{U}} \mathcal{E}\left(\int_{t_j}^{t_N} x'(t)N(t)x(t) + u'(t)M(t)u(t)dt + x'_{t_N}X_f x_{t_N}\right). \quad (8)$$

Following [14, 10] $V_j(\cdot)$ satisfies (7). Also it follows from (8), that if the stochastic control problem has a solution, then $V_0(x_0)$ exists and is finite.

Necessity. Equation (7) implies that for the controller associated with the switching sequence $\{i_j\}_{j=0}^{N-1}$ we have

$$\mathcal{E}\left(\int_0^{t_N} x'(t)N(t)x(t) + u'(t)M(t)u(t)dt + x'_{t_N}X_f x_{t_N}\right) \le V_0(x(0)). \quad (9)$$

Since $V_0(x(0)) = J(u^*, x(0)) \le J_{u \in \mathcal{U}}(u, x(0))$, the controller associated with the switching sequence, $\{i_j\}_{j=0}^{N-1}$, solves the state feedback stochastic control problem.

Suppose that the given state feedback controllers (2) are of the form

$$u_1(t) = K_1(t)x(t), \quad u_2(t) = K_2(t)x(t), \quad \ldots, \quad u_k(t) = K_k(t)x(t) \quad (10)$$

where $K_i(t)$ are given matrices of time, then we have.

Definition 2 Certainty Equivalence *[10, 6]. The system (1) is said to satisfy a certainty equivalence principle if the optimal admissible control law is independent of the statistics of the noise inputs.*

Corollary 1 *Consider system (1) and state feedback controllers (10). For this class of stochastic control problems certainty equivalence does not hold.*

Proof. Consider system (1) with the state feedback basic controllers (10). All that is required to be shown is that the controller switching sequence for the plant with no noise is different to that with noise.

The cost $J(u)$ for all admissible controllers can be rewritten as

$$J_{u \in \mathcal{U}}(u) = \mathcal{E}\left(x'_{t_N}X_f x_{t_N} + \int_0^{t_N} x'(t)(N(t) + K'_{i_t}R(t)K_{i_t})x(t)dt\right). \quad (11)$$

Expressing the state as

$$x(t) \triangleq \tilde{x}(t) + x^*(t) \quad (12)$$

with

$$\dot{\tilde{x}}(t) = (A + B_2 K_{i_t})\tilde{x}(t), \quad \tilde{x}(0) = x(0) \quad (13)$$

$$dx^*(t) = (A + B_2 K_{i_t})x^*(t)dt + dw(t) \quad x^*(0) = 0, \quad (14)$$

it follows that

$$\mathcal{E}x^*(t) = 0, \ \mathcal{E}\tilde{x}'(t)x^*(t) = \tilde{x}'(t)\mathcal{E}x^*(t) = 0 \ \forall t. \tag{15}$$

Following standard dynamic programming we substitute of (14) and (15) into (11) and rewriting (11) as:

$$J_{u \in \mathcal{U}} = \mathcal{E}\left(x'_{t_N}X_f x_{t_N}\right) + \mathcal{E}\left(\int_{t_{N-1}}^{t_N} \tilde{x}'(t)(N(t) + K'_{i_t}R(t)K_{i_t})\tilde{x}(t)dt\right) \tag{16}$$

$$+ \mathcal{E}\left(\int_{t_{N-1}}^{t_N} x^{*'}(t)(N(t) + K'_{i_t}R(t)K_{i_t})x^*(t)dt\right)$$

$$+ \mathcal{E}\left(\sum_{n=0}^{N-2}\int_{t_n}^{t_{n+1}} x'(t)(N(t) + K'_{i_t}R(t)K_{i_t})x(t)dt\right).$$

It can now be seen that in general the choice of gain matrix that optimizes the second and third terms in the above expression (16) will depend on both the state as well as the noise covariance and its contribution to the state over the interval $[t_{N-1}, t_N)$. This contradicts certainty equivalence.

Remark In switched controller systems the optimal controller switching sequence for linear feedback controllers depends on the input noise covariance. This result, also applies to zero-mean white Gaussian process noise inputs. This differs from optimal linear quadratic Gaussian control [7, 6, 15] were the optimal feedback gain is independent of the input noise covariance.

We illustrate that Corollary (1) above by considering the system (1) with parameters (29) in the simple scenario corresponding to only one switching time with switching occurring at $T = 0$. Let the system be defined over the interval $[0, 0.1)$ with $X_f = 0$. The process noise $w(t)$ is zero mean Gaussian with covariance $\mathcal{E}(w(t)w'(t)) = 1.0$. For this plant the cost (when there is no process noise is given by the second component of (16)) and is equal to $J(u_1) = x'_0 Z_1 x_0$ for controller u_1 and $J(u_2) = x'_0 Z_2 x_0$ for controller u_2 where

$$Z_1 = \begin{bmatrix} 0.5159 & -0.0314 \\ -0.0314 & 0.3257 \end{bmatrix} Z_2 = \begin{bmatrix} 0.5701 & -0.0873 \\ -0.0873 & 0.3459 \end{bmatrix}.$$

When there is noise the cost of using any of the given controllers is determined by the second and third components of (16) and is equal to $J(u_1) = x'_0 Z_1 x_0 + 0.0152$ or $J(u_2) = x'_0 Z_2 x_0 + 0.0210$. Consider the point $x = [0.1, 0.1]$ for the case without any noise, u_2 is the optimal control to use, whereas with noise the optimal control to use is u_1.

It can be clearly seen from Figures (1) and (2) that the controller switching regions depend on the process noise covariance. This differs from standard L.Q.G. optimal control were the controller gain matrix is independent of the input noise covariance for zero-mean Gaussian process noise.

3 Optimal Controller Switching with Partial Information

Let ξ be a random variable with Gaussian probability law, mean x_0 and covariance P_0 and $v(\cdot)$ is a Wiener processes independent of $w(\cdot)$ with covariance matrix $R(\cdot)$. Consider the following stochastic system whose dynamics on the interval $[0, t_N)$ are described by the stochastic differential equations

$$dx(t) = (A(t)x(t) + B_2(t)u(t))dt + B_1(t)dw(t), \quad x(0) = \xi,$$
$$dy(t) = C(t)x(t)dt + dv(t), \tag{17}$$

where $x(t) \in \mathbf{R}^n$ denotes the state, $y(t) \in \mathbf{R}^p$ denotes the output and $u(t) \in \mathbf{R}^h$ is the control input. The functions $A(t)$, $B_1(t)$, $B_2(t)$, $C(t)$ are known given piecewise continuous matrix functions of time. We assume that $R(t) \geq \rho I$, $\rho > 0$ $\forall t \in [0, t_N)$, and we define the following two families of sigma algebras

$$\mathcal{F}^t = \sigma(\xi, w(s), y(s), s \leq t)$$
$$\mathcal{Z}^t = \sigma(y(s), s \leq t) \tag{18}$$

Controller Switching with Output Feedback [12,5,4,13]. Given a collection of output feedback controllers

$$u_1(t) = K_1(t)y(t), \quad u_2(t) = K_2(t)y(t), \quad \ldots, \quad u_k(t) = K_k(t)y(t) \tag{19}$$

where $K_1(\cdot), K_2(\cdot), \ldots, K_k(\cdot)$ are given continuous matrix functions of time, we define a switching function $\Phi_j(\cdot)$ which is permitted to switch the controllers (19) at switching times t_j $\{j = 1, \ldots, N\}$. The class of output switching functions considered in this section is of the form $\Phi_j(\cdot) : \{\mathcal{Z}^{t_j}\} \to \{1, 2, \ldots, k\}$, which map from the output sigma algebra $\{\mathcal{Z}^{t_j}\}$ to the set of symbols $\{1, 2, \ldots, k\}$ which index controllers (19). This class of switching functions defines the family of dynamic nonlinear output feedback controllers of the form:

$$\forall j \in \{0, 1, \ldots, N-1\} \quad u(t) = K_{i_j}(t), y(t) \quad \forall t \in [t_j, t_{j+1}) \quad \text{where} \quad i_j = \Phi_j(\mathcal{Z}^{t_j}). \tag{20}$$

The control strategy now becomes a rule for switching from one controller to another to construct a symbolic sequence $\{i_j\}_{j=0}^{N-1}$ from the sigma algebra $\mathcal{Z}^{t_j}$. Furthermore let $\mathcal{X}$ denote the set of all controls achievable by controller switching (20) with the controllers (19) adapted to $\mathcal{Z}^t$.

The solution to the partial information stochastic control problem considered in this section involves the following Ricatti differential equation

$$\dot{P}(t) = P(t)A'(t) + A(t)P(t) - P(t)C'(t)R(t)^{-1}(t)C(t)P(t) + B_2(t)Q(t)B_2'(t)$$
$$\text{where } P(0) = P_0 \tag{21}$$

We consider the estimator equations of the form

$$\dot{\hat{x}}(t) = A(t)\hat{x} + P(t)C'(t)R(t)^{-1}(y(t) - C(t)\hat{x}) \text{ where } \hat{x}(0) = \xi \tag{22}$$

which correspond to the continuous time Kalman filter [16].

Let $S(\cdot)$ be a given function which maps from R^n to R and let $\hat{x}_0 \in R^n$ be a given vector. Introduce the following cost function:

$$W_1(t, \hat{x}, u(t)) \triangleq \mathcal{E}\left(\hat{x}'(t)N(t)\hat{x}(t) + u'(t)M(t)u(t)\right). \tag{23}$$

Then

$$F_j^i(\hat{x}_0, S(\cdot)) \triangleq \mathcal{E}\left(S(\hat{x}(t_{j+1}))\right) + \int_{t_j}^{t_{(j+1)}} W_1(t, \hat{x}, u_i(t), y(t))dt. \tag{24}$$

Definition 3 *The output feedback stochastic control problem with system (17) is said to have a solution via controller switching with the controllers (19) if the following conditions hold:*

(i) For any admissible control sequence, $u \in \mathcal{X}$, and initial condition $x(0)$ there exists a unique trajectory to the system state on the interval $[0, t_N)$ and the solution to the Ricatti differential equation (21) exists and is positive definite for all $t \in [0, t_N)$.

(ii) The cost given by

$$J(u)_{u \in \mathcal{X}} = \mathcal{E}\left(\int_0^{t_N} x'(t)N(t)x(t) + u'(t)M(t)u(t)dt + x'_{t_N} X_f x_{t_N}\right) \tag{25}$$

exists and is finite for some admissible control sequence, $u \in \mathcal{X}$,

Theorem 2. *The output feedback stochastic control problem (Definition (3)) has a solution via controller switching with the output feedback controllers (19) if and only if the solution $P(\cdot)$ to the Ricatti equation (21) with initial condition $P(0) = P_0$ is defined and positive definite on the interval $[0, t_N)$ and the dynamic programming equation*

$$V_N(\hat{x}_{t_N}) = \mathcal{E}\left(\hat{x}'_{t_N} X_f \hat{x}_{t_N}\right); \quad V_j(\hat{x}_{t_j}) = \min_{i=1,\ldots,k} \mathcal{E}\left(F_j^i(\hat{x}_{t_j}, V_{j+1}(\hat{x}_{t_{j+1}}))\right) \tag{26}$$

has a solution for $j = 0, 1, \ldots, N-1$ for all $\hat{x}_0 \in \mathbf{R}^n$.

Furthermore let $i_j(\hat{x}_0)$ be an index such that the minimum in (26) is achieved for $i = i_j(\hat{x}_0)$ and $\hat{x}(\cdot)$ be the solution to the equation (22) with initial condition $\hat{x}(0) = x_0$. Then the controller (19), (20) associated with the switching sequence $\{i_j\}_{j=0}^{N-1}$ where $i_j \triangleq \Phi_j(\hat{x}(t_j))$ solves the output feedback control problem.

Proof. (Outline) By virtue of Girsanov's theorem, Lemmata (1,2,3,4) in Section 6 below and Lemma (2.4.3) from [6] it follows that the evolution of the state of (17) with $u(t) \equiv 0$, denoted $\alpha(t)$ satisfies $\mathcal{E}^u[\alpha(t)|\mathcal{Z}^t] = \mathcal{E}^u[\alpha(t)|\mathcal{F}_u^t]$. Thus the states estimates generated by a Kalman filter are optimal when there is controller switching as considered in this paper. This is established in detail in section 6. It then follows that under the new probability measure $\mathcal{P}^u$ (defined in section 6), the estimation error, $\epsilon \triangleq x(t) - \hat{x}(t)$ is a Gaussian process with zero

mean and covariance matrix given $P(t)$ by (21). Hence, if the output feedback stochastic control problem has a solution via controller switching it follows that the solution, $P(\cdot)$, to the Ricatti differential equation (21) with initial condition, $P(0) = P_0$, is defined and positive definite on $[t_0, t_N]$. Now from Lemma (4) we have that the minimization of cost (25) is equivalent to minimizing (26) where $\hat{x}$ is the estimator state, $V(\hat{x}(t_N) = \mathcal{E}^u \hat{x}(t_N) X_f \hat{x}(t_N)$ is the cost at the final position and

$$d\hat{x}(t) = (A(t)\hat{x} + B_2(t)u(t))dt + P(t)C^{'}(t)R(t)^{-1}(y(t) - C(t)\hat{x})dt$$
$$\triangleq (A(t)\hat{x} + B_2(t)u(t))dt + d\nu(t) \tag{27}$$

where
$$\mathcal{E}^u[\nu(t)] = 0$$
$$\mathcal{E}^u[\nu^{'}(t)\nu(t)] = P^{'}(t)C^{'}(t)R(t)^{-1}C(t)P(t). \tag{28}$$

The statement of the Theorem now follows immediately from Theorem 1.

Remark Results for discrete time switched controller systems can be shown to trivially follow from the results presented in this paper.

4 Illustrative Example

We consider the 2 dimensional, unstable, non-minimum phase system

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1.25 & 1 \end{bmatrix}, B_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C(t) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}^{'} \tag{29}$$

Consider the system (17) the following set of output feedback controllers

$$u_1(t) = [-1, 2]\hat{x}(t), \quad u_2(t) = [3, -6]\hat{x}(t), \tag{30}$$

and permissible controller switching at time instants jT, $j = 0, 1, \ldots, 250$, $T = 0.05$ over the time interval $[0,12.5)$ seconds. It can be easily verified that each of the controllers in (30) is independently unable to stabilize the plant.

For simulation purposes set to $X_f = 10I$ and the matrix observer Ricatti equation (21) initial condition $P(0) = I$. The process and observation noises were also set to $\mathcal{E}(w(t)) = 0$, $\mathcal{E}(w(t)w^{'}(t)) = 0.1$, $\mathcal{E}(\nu(t)) = 0$ and $\mathcal{E}(\nu(t)\nu^{'}(t)) = 0.1$. The matrices $N(t)$ and $M(t)$ are $N(t) = 0.1$ and

$$M(t) = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}. \tag{31}$$

Using the results of Theorem 2 we see in Figure (3) that by switching the two unstable controllers the system state remains within an a small neighborhood about the origin. The control input to the system, is shown in Figure (4). The switching between the two controllers is evident.

5 Conclusions

In this paper we presented a solution to certain problems in switched controller design for stochastic dynamical systems. The main result is a separation theorem for partial information systems which can be used to convert the partial information stochastic control problem to a complete information stochastic control problem. We also show that certainty equivalence does not hold and for different noise statistics the optimal switching sequence changes. Finally we show that the optimal sequence of controllers can be determined by solving a dynamic programming equation. The results presented in this paper can be extended to the infinite time interval after some technical issues are taken into consideration.

6 Appendix

There is an intrinsic difficulty in output feedback stochastic control problems arising from the fact that the controller and hence the control depends on the observation. This difficulty has been solved for linear quadratic Gaussian stochastic control problem but it is not immediately clear that these results cover the class of switching systems considered in this paper. Thus in order to establish Theorem (2), we need to establish the existence of a new probability measure in which the output and all the admissible controls are independent. This then overcomes the difficulty described above. In order to establish the existence of such a measure we will need the following definitions and Lemmata. The argument here essentially follows [6] except that special attention must be given to the switching times were solutions to the equations describing the dynamics are not well defined.

Define the processes $\alpha(\cdot)$, $\beta(\cdot)$ by

$$d\alpha = A(t)\alpha dt + dw(t), \quad \alpha(0) = \xi \tag{32}$$
$$d\beta = C(t)\beta dt + dv(t), \quad \beta(0) = 0$$
$$\tag{33}$$

also define x_1, y_1 by

$$dx_1 = (A(t)x_1 + B_2(t)u(t))dt, \quad x_1(0) = 0 \tag{34}$$
$$dy_1 = C(t)x_1 dt, \quad y_1(0) = 0. \tag{35}$$

For any admissible control $u(t)$ we define

$$x(t) = \alpha(t) + x_1(t)$$
$$y(t) = \alpha(t) + y_1(t). \tag{36}$$

We then consider the following two processes [6]

$$\eta^u(t) = \exp\left[\int_0^t x'(t)C'(t)R(t)^{-1}dy - \frac{1}{2}\int_0^t x(t)'C(t)'R(t)^{-1}C(t)x(t)ds\right] \tag{37}$$
$$d^u v(t) = y(t) - \int_0^t C(t)x(s)ds \tag{38}$$

where the superscript u is to show the dependence of η^u on the the control $u(t)$.

The process η^u (38) satisfies the stochastic differential equation for all time $t \in [0, t_N) \neq t_j$ where t_j are the controller switching times

$$d\eta^u = \eta^u (x'C')R^{-1}dy, \quad \eta^u(0) = 1. \tag{39}$$

Furthermore we define $J^*(u)$ as

$$J^*(u(\cdot)) = \tag{40}$$
$$\mathcal{E}\left[\eta^u(t_N)\left(x'(t_N)X_f x(t_N) + \int_0^{t_N} \eta^u(t)x'(t)N(t)x(t) + u(t)'M(t)u(t)\right)dt\right].$$

Since the controls are piecewise linear (affine in the state) between the switching times on the interval $[t_j, t_{j+1})$ all that is required to be shown for the existence of a new probability measure $\mathcal{P}^u$ which makes the observation process independent of the control are that the cost, $J^*(u(\cdot))$ under the new measure is finite and that the Radon-Nikodym derivative [11]

$$\frac{d\mathcal{P}^u}{d\mathcal{P}} = \eta^u(t_N) \tag{41}$$

is a martingale. This is established in Lemmata (1,2,3) below which closely follow [6] but with modifications required to account for controller switching on a countable set.

Lemma 1. *There exists a constant, C_1, independent of the control such that*

$$\mathcal{E}\eta^u(t) \mid a(t) \mid^4 \le C_1 \tag{42}$$

Proof. Consider the following approximation of η^u,

$$\eta_\epsilon^u(t) = \exp\left[\int_0^t x_\epsilon' C'(t) R(t)^{-1}dy - \frac{1}{2}\int_0^t (x_\epsilon' C'(t) R(t)^{-1} C(t) x_\epsilon ds\right] \tag{43}$$

where

$$x_\epsilon(t) = \frac{x(t)}{(1 + \epsilon \mid x(t) \mid^2)^{1/2}}. \tag{44}$$

There exists a subsequence, denoted by ϵ, such that

$$\text{almost surely (a.s.) } \eta_\epsilon^u(t) \to \eta^u(t) \; \forall t. \tag{45}$$

It follows that

$$d|\alpha(t)|^4 = 4|\alpha(t)|^2\alpha'(t)\left[A(t)\alpha(t)dt + dw\right] + 2\left[\alpha(t)^2 trQ(t) + 2\alpha'Q(t)a\right]dt \tag{46}$$

and

$$d\eta_{\epsilon}^{u}(t)|a(t)|^{4} = 4\eta_{\epsilon}^{u}(t)|\alpha(t)|^{2}\alpha'(t)\left[A(t)\alpha(t)dt + dw\right] + \eta_{\epsilon}^{u}(t)|\alpha(t)|^{4}x_{\epsilon}'C'R^{-1}dy$$
$$+ 2\eta_{\epsilon}^{u}(t)\left[\alpha(t)^{2}trQ(t) + 2\alpha'Q(t)a\right]dt \; \forall t \in [0, t_N) \neq t_j. \qquad (47)$$

Upon integrating between $(0, t)$, using the smoothing property of integrals and taking the expectation it follows that

$$\mathcal{E}\eta_{\epsilon}^{u}(t)|\alpha(t)|^{4} = \mathcal{E}|\xi|^{4} + 4\mathcal{E}\int_{0}^{t}\eta_{\epsilon}^{u}(s)|\alpha(s)|^{2}\alpha'(s)A(t)\alpha(t)dt$$
$$+ 2\mathcal{E}\int_{0}^{t}\eta_{\epsilon}^{u}(s)\left[|\alpha(s)|^{2}trQ + 2a'Qa\right]ds. \qquad (48)$$

Using the fact that $\mathcal{E}\eta_{\epsilon}^{u}(s) = 1 \; \forall s$ it can be shown that

$$\mathcal{E}\eta_{\epsilon}^{u}(t)|\alpha(t)|^{4} \leq \mathcal{E}|\xi|^{4} + C_{2}\left(1 + \mathcal{E}\int_{0}^{t}\eta_{\epsilon}^{u}(s)|\alpha(s)|^{4}ds\right) \qquad (49)$$

where C_{2} is a constant independent of ϵ and $u(\cdot)$. Using the Gronwall Lemma one immediately obtains

$$\mathcal{E}\eta_{\epsilon}^{u}(t)|\alpha(t)|^{4} \leq C_{1}. \qquad (50)$$

Hence taking the limit as $\epsilon \to 0$, the claim of the lemma immediately follows.

Lemma 2. *For any admissible control, $u \in \mathcal{X}$, and for all trajectories the cost denoted by $J^{*}(u(\cdot))$ is finite.*

Proof. It follows from the Gronwall Lemma that

$$|x_{1}(t)| \leq C_{3}\left(\int_{0}^{T}|u(\cdot)|^{2}ds\right)^{1/2}. \qquad (51)$$

Using the definition of (36) and from Lemma (1) it follows that on the finite interval $[0, t_N)$ there exists a constant C_{1} such that

$$\mathcal{E}\eta_{\epsilon}^{u}(t)|\alpha(t)|^{2} \leq C_{1}. \qquad (52)$$

substituting into $J^{*}(u(\cdot))$ the claim of the lemma is established.

Lemma 3. $\eta^{u}(t)$ *is a $\mathcal{F}^{t}$ martingale.*

Proof. From Lemma 2.4.2 (pg 40) of [6] all that needs to be established is that

$$\mathcal{E}\int_{0}^{t_N}\eta^{u}(t)|u(t)|^{2}dt < \infty. \qquad (53)$$

Using the fact that the system is piecewise linear and $\eta^{u}(t)$ is finite for all $t \in [0, t_N)$ it follows trivially that there exists a constant C_{3} such that

$$\mathcal{E}\int_{0}^{t_N}\eta^{u}(t)|u(\cdot)|^{2}dt < C_{3}. \qquad (54)$$

The following Lemma is also needed in the proof of Theorem (2).

Lemma 4. *Suppose that the solution $P(\cdot)$ to the Ricatti differential equation (21) with initial condition $P(0) = P_0$ is defined and positive definite on the interval $[0, t_N)$ then*

$$\min_{i=1,\ldots,N} \mathcal{E}^u \left(x^{'}(t_N) X_f x(t_N) + \int_0^{t_N} x^{'}(t) N(t) x(t) + u'(t) M(t) u(t) dt \right)$$

$$= \min_{i=1,\ldots,N} \mathcal{E}^u \left(\hat{x}^{'}(t_N) X_f \hat{x}(t_N) + \int_0^{t_N} \hat{x}^{'}(t) N(t) \hat{x}(t) + u'(t) M(t) u(t) dt \right)$$

$$+ \mathcal{E}^u \left(\epsilon^{'}(t_N) X_f \epsilon(t_N) + \int_0^{t_N} \epsilon^{'}(t) N(t) \epsilon(t) dt \right). \tag{55}$$

Proof. Rearranging (25) we have

$$\mathcal{E}^u \left(\mathcal{E}^u \left[x^{'}(t_N) X_f x(t_N) + \int_0^{t_N} x'(t) N(t) x(t) + u'(t) M(t) u(t) dt \mid \mathcal{Z}^t \right] \right) =$$

$$\mathcal{E}^u \left(\mathcal{E}^u \left[(x(t_N) - \hat{x}(t_N))^{'} X_f (x(t_N) - \hat{x}(t_N)) + (x(t_N) - \hat{x}(t_N))^{'} X_f \hat{x}(t_N) \mid \mathcal{Z}^t \right] \right)$$

$$+ \mathcal{E}^u \left(\mathcal{E}^u \left[\hat{x}^{'}(t_N) X_f (x(t_N) - \hat{x}(NT)) + \hat{x}^{'}(t_N) X_f \hat{x}(t_N) \mid \mathcal{Z}^t \right] \right) +$$

$$\mathcal{E}^u \left(\mathcal{E}^u \left[\int_0^{t_N} (x(t) + \hat{x}(t))^{'} N(t) (x(t) - \hat{x}(t)) + \right. \right.$$

$$(x(t) - \hat{x}(t))^{'} N(t) \hat{x}(t) - \hat{x}^{'}(t) N(x(t) - \hat{x}(t)) \mid \mathcal{Z}^t \right] \Big) +$$

$$\mathcal{E}^u \left(\mathcal{E}^u \left[\int_0^{t_N} \hat{x}^{'}(t) N(t) \hat{x}(t) + u'(t) M(t) u(t) dt \mid \mathcal{Z}^t \right] \right). \tag{56}$$

Using the orthogonality property of the Kalman filter [16], namely $\mathcal{E}[(x-\hat{x}) Z \hat{x}^{'}] = 0$ where Z is any matrix, it immediately follows that

$$\mathcal{E}^u \left(\mathcal{E}^u \left[x^{'}(t_N) X_f x(t_N) + \int_0^{t_N} x^{'}(t) N(t) x(t) + u'(t) M(t) u(t) dt \mid \mathcal{Z}^t \right] \right)$$

$$= \mathcal{E}^u \left(\mathcal{E}^u \left[\hat{x}^{'}(t_N) X_f \hat{x}(t_N) + \int_0^{t_N} \hat{x}^{'}(t) N(t) \hat{x}(t) + u^{'}(t) M(t) u(t) dt \mid \mathcal{Z}^t \right] \right)$$

$$+ \mathcal{E}^u \left(e^{'}(t_N) X_f e(t_N) + \int_0^{t_N} e^{'}(t) N e(t) dt \right). \tag{57}$$

The result now follows immediately.

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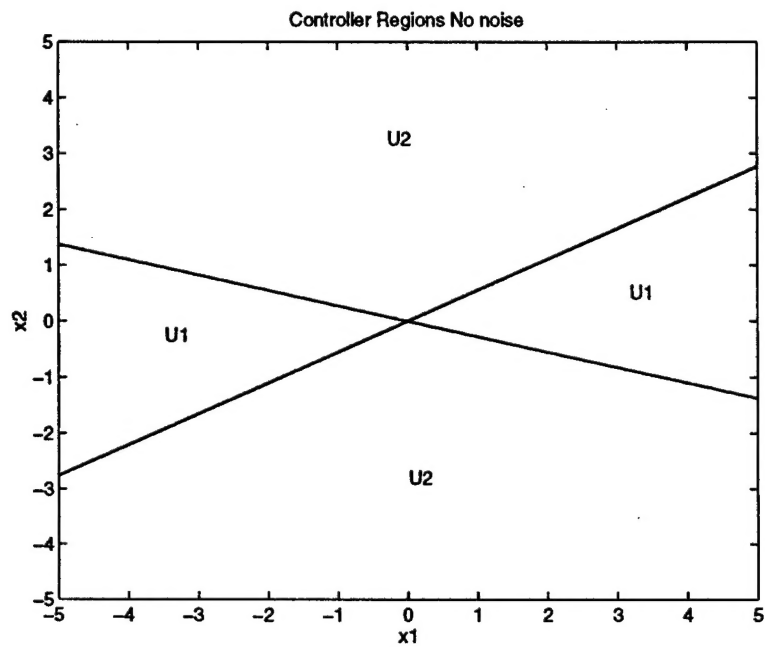16. B.D.O Anderson and J.B. Moore. *Optimal Filtering*. Prentice Hall, Englewood Cliffs, N.J., 1979.

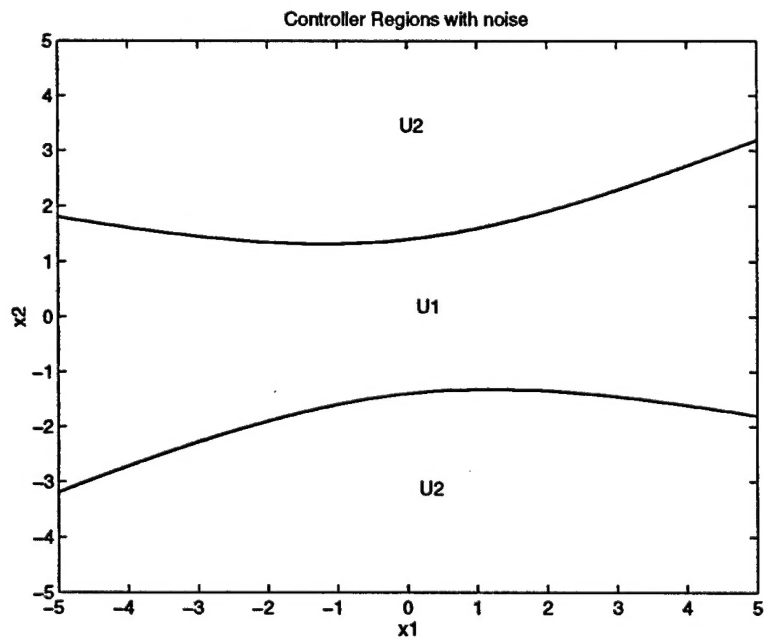**Fig. 1.** Controller decision diagram with no noise.

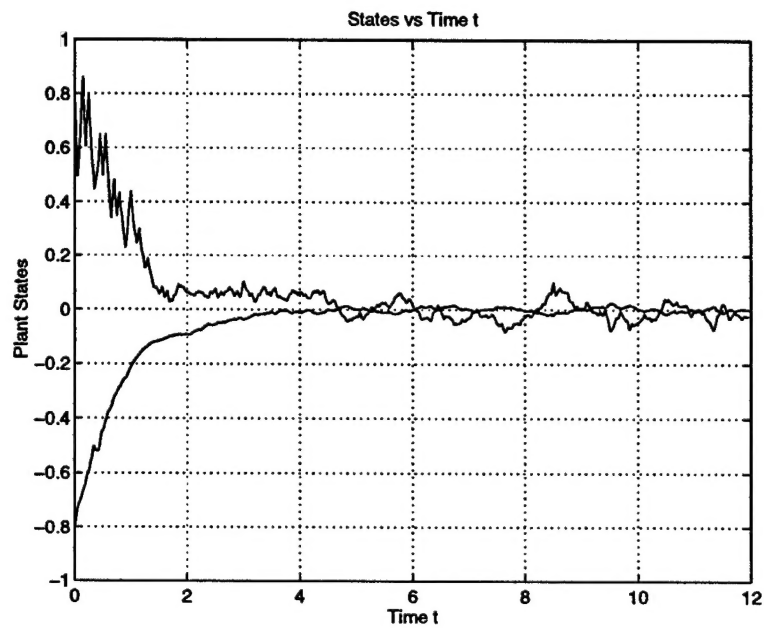**Fig. 2.** Controller decision diagram with noise, $E[w] = 0, E[w'w] = 1.0$

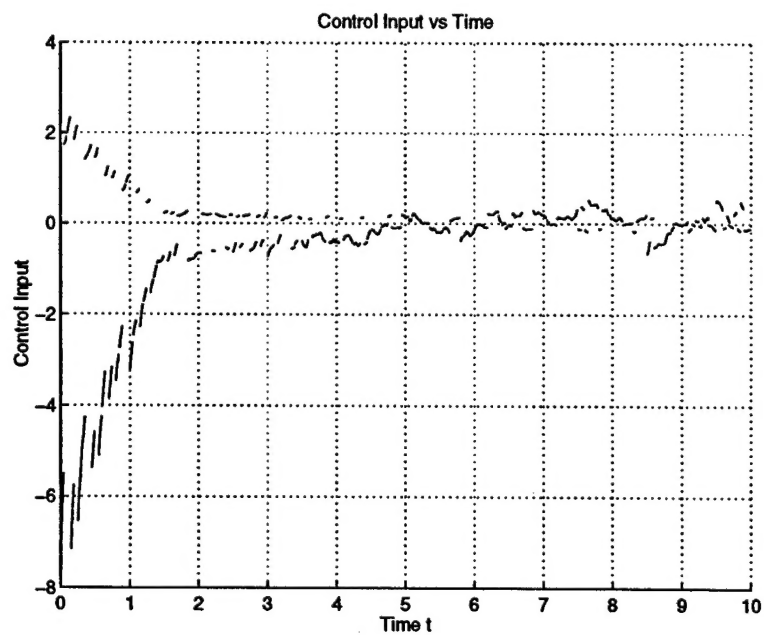**Fig. 3.** Evolution of plant states with time.

Fig. 4. Evolution of control input u(t) with time for the switching controller.